Introduction to the Semiology of Mathematical Practices

Juan Luis Gastaldi

Abstract The philosophy and history of mathematical practices have brought the study of mathematical language and signs to the forefront of contemporary mathematical thought. However, despite the fruitfulness of this research trend, a comprehensive and unified account of its various aspects and the diverse approaches taken to explore it remains elusive. Recognizing this gap, we have undertaken the task of editing the present section of the Handbook of the History and Philosophy of Mathematical Practice as a much-needed remedy. Before providing an overview of the various contributions to the section, this introduction provides some context for the subject matter and a few conceptual clarifications.

Key words: Semiology, Semiotics, Signs, Peirce, Saussure, Mathematical Practice, Language, Textuality

Introduction

The purpose of the following pages is to present the section "Semiology of Mathematical Practices" in this Handbook of the History and Philosophy of Mathematical Practice (Sriraman 2020). The section provides a comprehensive account of the philosophical and historical approaches to the study of mathematical *signs* and *language* from the viewpoint of mathematical practice. Conceived as a whole, the series of contributions that make up this section demonstrate the consistency and diversity of these approaches, offering a unified yet plural perspective on this important dimension of the history and philosophy of mathematical practices.

The section is composed of nine chapters written by leading specialists, providing a rich account of the state of the art in most areas of this research field. The initial

Juan Luis Gastaldi
ETH Zürich, Clausiusstrasse 59, 8092 Zürich, e-mail: `juan.luis.gastaldi@gess.ethz.ch`

chapters present the philosophical and historical underpinnings of a practice-driven theory of mathematical signs. They are followed by compelling case studies, which demonstrate the fruitfulness of the semiological approach in analyzing the historical and conceptual aspects of mathematical practices. Finally, the section offers fresh insights into current and upcoming challenges where a theory of mathematical signs can play a critical role.

The authors of this section have done an exceptional job of enriching their unique perspectives with a comprehensive introduction to the respective topics, along with a presentation of the necessary context and conceptual tools. Consequently, attempting to provide a detailed introduction to the semiology of mathematical practices here would lead to redundancy. Instead, the goal of these pages is to offer a concise presentation of all the contributions, accompanied by a few elements of context and some conceptual clarifications that could only be addressed from the general standpoint enabled by the completed work.

What Do You Mean, "Semiology"?

An initial clarification is in order concerning the term "semiology". While some may find the term either unknown or outdated, it addresses a precise object of inquiry that no other alternative seems to capture entirely. At its core, semiology is the *theory of signs*. The study of signs has a rich history, stretching back to ancient medical practices where signs and symptoms were essential in diagnosing ailments. Building upon this foundation, the Stoics further developed the concept of the sign within the context of logical thought. In modern philosophy, John Locke drew upon this tradition to characterize one of the three divisions of science as "σημειωτική" (*semeiotiké*), concerned with "the nature of signs the mind makes use of for the understanding of things, or conveying its knowledge to others" (Locke 2013, p. 720). In contemporary thought, semiology has two primary roots that have significantly shaped its present understanding. One of these roots can be traced back to Charles Sanders Peirce within the tradition of Boolean logic. Peirce retained the term "semiotics" or "semeiotics" to refer to the philosophical study of the semiosis process, whereby a sign, a signified object, and an interpretant are related. Parallelly, within the linguistic tradition, Ferdinand de Saussure introduced the term "semiology" as a key component of the ideas leading to the structuralist revolution in the field. At the opening of his influential *Course in General Linguistics* (Saussure 1959), Saussure stated:

> *A science that studies the life of signs within society* is conceivable; it would be a part of social psychology and consequently of general psychology; I shall call it *semiology* (from Greek *sēmeîon* 'sign'). Semiology would show what constitutes signs, what laws govern them. Since the science does not yet exist, no one can say what it would be; but it has a right to existence, a place staked out in advance. Linguistics is only a part of the general science of semiology; the laws discovered by semiology will be applicable to linguistics, and the latter will circumscribe a well-defined area within the mass of anthropological facts. (Saussure 1959, p. 16)

As Saussure explains in the passages following these lines, postulating semiology as a scientific discipline was crucial for successfully "assigning linguistics a place among the sciences" (p. 16). The reason is that a semiological approach to language seeks to determine the "true nature of language" by examining its similarities and differences with other sign systems, rather than deferring the task to the study of individual behaviors or social institutions. In this way, semiology affords the theory of language independence from fields such as psychology or sociology.

It is this specific feature that our use of the term "semiology" would like to emphasize.[1] More precisely, our choice seeks to highlight the idea that there exists a consistent region of phenomena akin to language, which cannot be fully understood by relying solely on current linguistic analysis without missing crucial aspects. Yet, attempting to approach those phenomena from extra-linguistic perspectives can lead to even worse reductions to the principles of other disciplines such as psychology, sociology, logic, biology, physics, or even mathematics. If we agree to call "*signs*" those non-linguistic entities with language-like characteristics, the term "semiology" refers to the idea that these signs, irrespective of their connections to other domains of knowledge, are *things* that deserve independent and thorough study *in and of themselves*.

A less pedantic way of expressing the same idea is that the perspective adopted here considers semiology as a form of *generalized linguistics*, encompassing languages other than natural language. Hence its relevance to addressing the problem of what is usually referred to, for lack of a better expression, as *mathematical language*. Throughout its documented history, mathematical practice has employed all kinds of semiotic artifacts and principles to express the specific contents at stake. Those artifacts involved natural language in significant ways. Yet, crucially, they also transcended those resources in an equally essential manner. In its minimal expression, the idea of a semiology of mathematics is that all those expressive means and their relationship to their respective contents should be studied *as such* from a perspective that leaves room to explore the similarities and differences with other linguistic phenomena.

A second point of clarification concerns the distinction between the terms "semiology" and "semiotics". While these terms are often used interchangeably, our focus on the idea of a generalized linguistic in this section motivates a slight preference for "semiology", as evidenced by our choice of the section title. Other arguments stemming from the evolution of the structuralist tradition could be also summoned. In particular, Louis Hjelmslev, following Saussure's program, offered explicit definitions precisely characterizing the relation between the two notions. In Hjelmslev (1975), he provides the following definitions for both terms:

> A *Semiotic* [...] is a Hierarchy, any of whose Components admits of a further Analysis into Classes defined by mutual Relation, so that any of these classes admits of an analysis into Derivates defined by mutual Mutation. (Df. 24, p. 11)
>
> A *Semiology* [...] is a Metasemiotic that has a Non-Scientific Semiotic as an Object Semiotic (Df 47, p. 15)

[1] For a more comprehensive approach, the reader can consult Barthes (1988); Sebeok (2001); Robering et al (1996).

Though Hjelmslev's definitions may appear obscure—in particular, each capitalized expression has been previously defined in his treatise—his idea is simple: the term "semiotic" is reserved to refer to any sign *system*, while "semiology" denotes those sign systems that take other semiotics as their objects of analysis. These definitions have at least the merit of replacing the alternative between both terms with a joint and coherent characterization. In our case, this would mean characterizing mathematics as a semiotic, and our theories about mathematical signs as a semiology.[2]

However, although these and other arguments could be put forth to justify the choice of either term, it is crucial to keep in mind that our intention from the outset of this project was to take a minimalistic and non-prescriptive stance. Above all, our primary aim for this section was to enable a pluralistic approach to the study of mathematical language and signs, granting authors the liberty to expose their unique perspectives on this fundamental aspect of mathematical practices. Accordingly, we intentionally refrained from any standardization in the use of the terms "semiotics" and "semiology", nor did we encourage specific conceptual choices or theoretical preferences that the use of one term or the other might implicitly entail. Hence, as a rule, both terms are to be read as synonyms across the different chapters of this section, and any variations should be attributed to the authors' reflective choices or individual tastes.

Notwithstanding the plurality of perspectives represented in this section, there is one significant feature setting the semiological standpoint apart from other philosophical approaches to language, such as logicism or generative linguistics, which all our authors seem to share to some extent. This crucial aspect is the key role attributed to *practices* in the study of signs. This will come as no surprise in a Handbook on the history and philosophy of mathematical practices. However, from a semiological viewpoint, the relationship between signs and practices is not an accidental feature. Given its significance, it might be helpful to provide some conceptual insight on this point.

The emphasis on practices is a component of both the Peircian and the structuralist traditions. However, it is the latter that has given it a more prominent role, as evidenced by the development of Saussure's semiological project within various disciplines such as anthropology, psychology, sociology, and history, each one addressing specific dimensions of human practices. Within this tradition, the notion of practice is consubstantial to that of sign. We have seen how Saussure's semiological project placed language as part of "the life of signs within a society", determining "a well-defined area within the mass of anthropological facts" (Saussure 1959, p. 16). Yet, a more systematic account of the internal connection between signs and practices is provided by Hjelmslev and Uldall. In Hjelmslev and Uldall (1936), the authors write:

[2] The fact that semiologies are defined as having "non-scientific" semiotics as objects should not lead us to assume that mathematics cannot be studied as a science from a semiological perspective. For Hjelmslev, a "scientific semiotic" is merely a semiotic organized as a specific analytical procedure. If mathematics were to fall into this category, its study as a sign system would be termed a "meta-(scientific semiotic)" according to Hjelmslev's view (cf. Hjelmslev 1975, Df. 41,46).

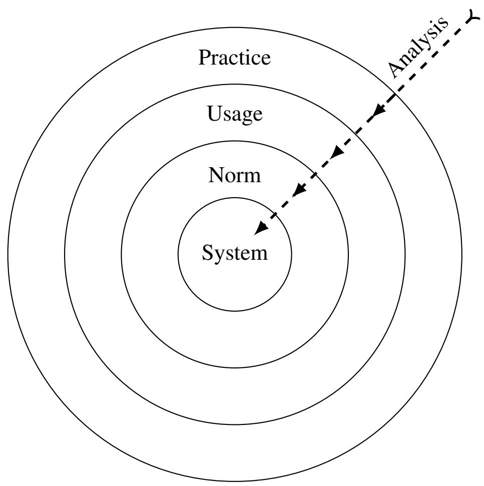

Fig. 1 Illustration of Hjelmslev and Uldall's (1936) view on the semiological analysis through the various dimensions of language.

> A language consists of three concentric parts [...]: A central part, the *system*, i.e. the elements arranged in a pattern of mutual relations; the *norm*, i.e. a set of rules based on the system and fixing the limit of variability for each element; the *usage*, i.e. a set of rules based on the norm and fixing the limit of variability tolerated in a given community at a given time. It is necessary to distinguish between language as an institution, and the use to which it is put by the individual speaker; this is called the *practice* (Saussure's *parole*). (Hjelmslev and Uldall 1936)

Figure 1 offers a possible illustration of Hjelmslev and Uldall's (1936) views. In their text, the authors propose the example of the English phoneme /r/, which, as an element of the English linguistic *system*, is defined through a complex array of oppositions with respect to all the other phonological units of the system. Yet, as well specified as this phoneme may be, it leaves significant room for variability within those limits, thus defining a *norm* for English determined by all the possibilities of pronouncing the letter *r* without it being mistaken for another element of the phonetic system. Specific *usages* of English, such as the Scottish, Victorian, or Jamaican, exploit only a restricted region of that norm. Finally, individual *practices* represent singular manifestations of those collective usages, like the unmistakable accent of Bertrand Russell. These conceptual distinctions can be easily imported into the analysis of mathematical language. An unimaginative example is provided by arithmetic, where numbers, defined as elements of a system, can be expressed in a myriad of ways without violating the elementary systematic requirements. However, different historical and geographic numeral systems (Roman, Arabic, binary, etc.) restrict the variability of that norm to only a few writing usage principles, which will then be practiced by individuals in their own style.

Interestingly, Hjelmslev and Uldall affirm that the true object of a theory of language and signs is the exhaustive description of a language *as a system*. However, the key point is that the elements and rules composing semiotic systems *are not observable as such*. In their most fundamental nature, *languages and signs are only given in individual practices*. As long as the analysis of signs is to remain empirical, its task is to infer systematic features from idiosyncratic and constantly evolving practices. As the Uldall and Hjelmslev say:

> "The [...] system is found inductively through a series of ascending abstractions: an empirical study of the practice leads to the recognition of the usage, a study of the usage to the recognition of the norm, and a study of the norm to the recognition of the system." (Hjelmslev and Uldall 1936)

A semiology of mathematics is, therefore, inseparable from the analysis of mathematical practices, even as it aims to reveal the underlying system. Nonetheless, a semiological approach must also characterize the distinctiveness of mathematics compared to all other sign systems. In this regard, it is not impossible that part of what makes mathematical practice different from most semiotic practices is its recurrent endeavor to explicitly articulate systematic features of its own language. Admittedly, these features themselves are conveyed through signs and are, as such, the object of individual practices. Peano axioms can certainly be taken as a characterization of the *system* of arithmetic. But this cannot erase the fact that such a system is the result of multiple systematization practices (including but not restricted to Peano's), involving a complex interplay with norms and usages. What is more, the stabilization of one system cannot entirely preclude the emergence of new and non-entirely-equivalent systematizations in the future. The chapters in this section provide numerous examples of this phenomenon. However, the explicit pursuit of systematization as a critical aspect of mathematical practices can motivate alternative original trajectories in the semiotic space, distinct from the ones traditionally conceived for the study of other sign systems. These and other specificities of mathematical practices indicate the potential contribution that a semiology of mathematics can offer to a general theory of signs.

Semiology in History and Philosophy of Mathematics: Contributions to This Section

In 1902, Bertrand Russell reproached Frege for not having "clearly disentangled the logical and linguistic elements of naming" (Russell 2010, p. 519). Half a century later, the young Noam Chomsky opposed any attempt to understand natural language as logical systems in the way of Carnap:

> But in the case of the artificial 'languages' investigated by Carnap in his logical laboratory, there is little if any antecedent reason for regarding these as in any way comparable to the actual languages of the outside world.(Chomsky 1955, p. 43)

These cross-accusations involving the most prominent figures in the fields od logic and linguistics serve as a compelling illustration that despite the so-called "linguistic turn", insofar as the philosophy of mathematics has been governed by a classic logical approach, the true problems of language have been kept at arm's length. The tutelary figure of the late Wittgenstein only confirms this circumstance, as it demonstrates how a multitude of new problems linked to mathematical signs emerge once the strict logical perspective is subject to critical examination.

As a consequence, strictly semiological approaches have occupied a relatively marginal position in contemporary philosophy of mathematics. As the new century emerged, alternative approaches to the language of mathematics, distinct from strictly logical accounts, were limited to only a few rather disconnected and not necessarily converging individual works originating from both philosophy and history (cf.,for instance, Rotman 1988, 2000; Netz 1999; Vinciguerra 1999; Herreman 2000; Serfati 2005; Kvasz 2008; Macbeth 2005; Chemla 2004). However, as a result of the "practical turn" in the philosophy of mathematics over the past decades, numerous studies have been pursued to explore various facets of mathematical knowledge as products of human practices rather than solely arising from abstract logical properties. Accordingly, a significant portion of research within the field involves the analysis of various mathematical corpora. These corpora encompass a wide range of materials, including actual mathematical texts, experimental data, historical and contemporary works, whether authored individually, institutionally, or collectively, and comprising both published and unpublished sources. The problem of mathematical signs—inscriptions, symbols, marks, diagrams, expressive means, representations, etc.—has thus regained prominence in the field. Drawing on insights from classical semiological theories, important aspects of mathematical knowledge have been shown to be intricately related to regularities and emergent patterns identifiable at the level of signs. From a philosophical viewpoint, these works have contributed to bridging the traditional gap that separated mathematical from natural language by demonstrating the manifold ways in which mathematical signs, classically assumed to be purely formal, are conditioned by empirical and historically determined human practices. This shift in perspective has further facilitated the avoidance of anachronistic and ethnocentric biases in the philosophical and historical study of mathematical knowledge.

The philosophy and history of mathematical practices have thus brought the study of mathematical language and signs to the forefront of contemporary mathematical thought. However, despite the fruitfulness of this research trend, a comprehensive and unified account of its various aspects and the diverse approaches taken to explore it remains elusive. Recognizing this gap, we have undertaken the task of editing the present Handbook section as a much-needed remedy.

As emphasized earlier, our primary aim was to prioritize the richness of diverse perspectives rather than impose artificial uniformity and systematicity in the practice-driven study of mathematical language. To achieve this goal, we sought to include a wide array of viewpoints and topics in our selection of contributions. Despite the varied nature of approaches, the resulting series of chapters exhibits a cohesive structure. The section commences with papers that tackle the problem of mathematical signs from a broad and overarching perspective, both philosophically and historically. These pieces set the stage for the exploration of case studies that vividly demonstrate the fruitfulness of a semiological approach in analyzing various historical and conceptual dimensions of mathematical practices. Finally, the section concludes by offering fresh perspectives that confront new challenges, paving the way for further developments that can build upon and expand the rich insights showcased in this section.

The first chapter of this section is **David Waszek**'s *Signs as a Theme in the Philosophy of Mathematical Practice*. This contribution is conceived as a detailed introduction to the whole section. Waszek provides a comprehensive *survey* of the diverse works that delve into the problem of mathematical signs within the tradition of the history and philosophy of mathematical practice. His insightful analysis explores the reasons behind this apparent convergence in the field towards the study of signs, revealing that it conceals a diversity—and sometimes even a disparity—of perspectives and motivations, each with its own agenda.

Waszek identifies three main trends within this landscape. The first trend aims to reveal the norms of informal mathematical practices, suggesting that mathematics is characterized by specific manipulations of signs rather than solely relying on classic logical inference. The second trend focuses on exploring the boundaries of rigorous norms, such as ambiguity and polysemy, to underscore the openness of mathematical practices to socio-historical conditions. Lastly, the third orientation investigates the cognitive aspects associated with the manipulation of signs by individuals, raising questions about the cognitive conditions that shape mathematical knowledge and the design choices that can enhance or hinder the efficiency of mathematical practices.

Notably, the three trends identified by Waszek align, at least to some degree, with the three external layers of semiotic systems presented earlier: abstract norms, socio-historical usages, and individual practices. One can hardly conceive of a better account of how the current research on the semiology of mathematical practices is distributed across the landscape of a theory of signs, encompassing multiple trajectories.

After Waszek's comprehensive survey, **Roy Wagner** proposes a more theoretical introduction to the topic in the chapter titled *Structural semiotics as an ontology of mathematics*. Wagner's specific approach revolves around the *structuralist and post-structuralist theories of signs*, building upon the foundational works of Saussure, Lévi-Strauss, and Derrida. His analysis conceptually reconstructs the historical evolution of this major current of semiological thought, skillfully extracting the elements and principles relevant to the study of mathematical practices. In particular, Wagner centers his account around the *commutation test*, which allows to identify semiotic units through correspondences between signifier and signified substitutions. Wagner's reconstruction is both precise and accessible, making the chapter an exceptional resource for readers seeking an initial foray into the semiology of mathematics.

However, Wagner's contribution goes beyond the sole presentation of this theoretical framework; he also puts the corresponding conceptual tools to the test with a historically informed mathematical example—the history of the numeral 0. This example not only showcases the strengths of the approach but also highlights its limitations. Through this elaboration, Wagner advances a crucial thesis: the semiological perspective in question can offer more than just methodological tools or epistemological insights to the history and philosophy of mathematics; *it can offer an ontology*. In other words, the analysis of mathematical signs involves a way of answering the question of what mathematics *is* or what it is *about*.

Crucially, Wagner avoids falling into a naïve position holding that everything is a sign. Instead, he crafts a formula that is rich with subtleties: everything is *also* a sign. This implies that everything, including everything mathematics is supposed to be about, is at any time subject to the displacement operations necessary to determine it as a sign through the commutation test. Departing from classic ontologies of mathematics centered around fixed referents and concepts, Wagner concludes, supported by various examples, that this perspective offers an insightful image of mathematics as a series of signs determinable only through the regularities found amidst multiple changes of context. Consequently, mathematical knowledge becomes inherently reliant on practices in a central and inescapable manner.

Moving to the next chapter, we are reminded that the role of history in fostering a renewed awareness of the significance of signs for the study of mathematical practices cannot be overstated. Notably, Reviel Netz's *The Shaping of Deduction in Greek Mathematics* Netz (1999) stands as a landmark and an exemplar in this regard. The reader will certainly recall Netz's celebrated "two lanes" thesis, positing that the two components of deduction in Greek mathematics—necessity and generality—were shaped through a complex articulation between two semiotic devices: the lettered diagram, and the formulaic language. While the semiotic aspects of these "cognitive tools" were expounded with remarkable detail and erudition, the methodological and epistemological underpinnings of the analysis ended up appearing scarce in the face of the increasing fruitfulness of the approach.

The significant impact of this historical study led to a growing need for further theoretical and methodological elaboration guiding the replication of the same historiographical gesture in other contexts. In the third chapter of this section, **Reviel Netz** willingly embraced the challenge of revisiting his celebrated work after almost a quarter of a century since its publication. In *Shaping, Revisited*, Netz embarks on a remarkable endeavor of retroactive methodological and epistemological reflection, directly addressing the semiological dimension of this and subsequent works. Reviewing a rich series of historical cases, he probes the attributes that can be attributed to strictly semiotic properties in each case. This reflective exercice reveals that the study of mathematical signs, from a strictly semiological perspective, is inextricably intertwined with two other dimensions: the "culturally and historically malleable" cognitive apparatus and, more crucially, a materialist approach to the history of culture. Thus, we see the name of Marx joining that of Lévi-Strauss to indicate the plausibility of a materialist-structuralist perspective in the history of mathematics, yet only as a provisional orientation contingent on the empirical conditions of particular historical explanations.

The intricate connection between textual practices and culture in the historical study of mathematical practices has been a focus of an intense research program led by Karine Chemla and her colleagues for over two decades Chemla (2004). In the fourth chapter of this section, titled *Mathematical Practices and Written Evidence:*

General Reflections Based on a Historian's Experience, **Karine Chemla** returns to this question through a theoretical lens, offering critical insights into the *nature of textuality* in the history and philosophy of mathematical practice.

Beginning with a clear distinction between texts, documents, and inscriptions, Chemla builds on a comprehensive survey of recent works on the topic, drawing from a historically grounded theoretical discussion on how these forms of textual evidence can inform our understanding of mathematical practice. She first challenges the prevailing notion that the primary function of texts is communication. While this may be true for many published texts, historical investigations reveal that communication is not an immutable property of textuality but rather a situated practice that demands proper contextualization. Yet, the universality of communication can also be questioned at a deeper level, particularly when the notion of textuality is extended beyond published sources. Historical records offer numerous instances where texts serve other purposes than communication, such as exploration, learning, or computation.

By meticulously surveying these diverse cases, in which cultural variability plays a pivotal role, Chemla convincingly demonstrates how different conceptual and philosophical perspectives on the nature of writing can shape our understanding of texts. Emphasizing the importance of looking beyond published, communicative documents and considering the materiality of texts, she reveals a wealth of mathematical practices and sheds light on the social conditions that enabled their existence.

The first four chapters of the section offer an overarching perspective on how philosophical considerations about signs and language can ground the analysis of concrete, historical mathematical practices, while simultaneously showcasing how historical analyses of practices can inform and shape philosophical perspectives. With **Lucien Vinciguerra**'s contribution, *What happens, from a historical point of view, when we read a mathematical text?*, the section takes a step further by presenting a series of three contributions that exemplify this virtuous circle in action, combining conceptual elaborations with detailed case studies.

Vinciguerra's chapter is a testament to his rich and original contributions, where he extends Foucault's archaeological program to the history and philosophy of mathematics (Vinciguerra 1999, cf., in particular). Here, he provides a compelling example of how such a perspective can yield critical tools for a theory of mathematical signs, emphasizing the inseparability of philosophy and history in this context. The central focus is on the act of *reading* mathematical texts, which becomes pivotal in addressing the classic Husserlian question concerning the stability of mathematical truth within an ever-evolving history of mathematical objects and concepts.

Intriguingly, as entangled as philosophy and history may be, the nature of mathematics itself introduces a tension between both, which Vinciguerra embraces as productive without the need of taking a definitive stance. He proposes to view texts as continuous but divisible spaces, where partitionings are effectively enacted through reading practices. Specifically, Vinciguerra explores three distinct ways in which the act of reading can redistribute the lines within a text: partitioning what is true and false, partitioning what is text and what is image, and partitioning what is mathematical and what is not in a text.

Mobilizing these critical tools, the second part of the chapter presents a meticulous case study centered around the reading of equations in Descartes' *Geometry*, skillfully drawing insights from other Cartesian texts like *The World* and the *Dioptrics* to reveal hidden partitioning lines. Vinciguerra's original focus on the philosophical conditions of reading historical texts uncovers essential differences between the historicity of mathematics and that of experimental sciences.

Moving forward in history, **Ladislav Kvasz** proposes a stimulating perspective in his chapter, *Symbolic algebra as a semiotic system*, by delving into the *dynamic effects* that sign systems can exert on mathematical practices. Building on his prior work (Kvasz 2008), which associates crucial dimensions of the evolution of mathematical knowledge with different features of mathematical language, Kvasz's contribution focuses on the invention of symbolic algebra during the 16th and 17th centuries. He contends that understanding the changes occurring during this period requires attention to two dimensions of algebraic symbols: compositionality and referentiality, which characterize pivotal components of any semiotic system. To elucidate both aspects, Kvasz draws upon the works of Frege and Wittgenstein, respectively. Both references are surprising in their own way, determining the originality of Kvasz's approach. The first one because, unlike classic analytic approaches, Frege is here summoned to study the functioning of signs, and not of logical properties. Kvasz's use of Wittgenstein's philosophy, on the other hand, is doubly innovative. At odds with the usual references to the late Wittgenstein in this area of research, Kvasz returns to the early Wittgenstein. Yet, instead of relying on the *Tractatus*'s logical apparatus, the referential dimension of mathematical language is elaborated through the pictorial theory of signs. The result is a thought-provoking chapter proposing a framework wherein analytical properties of mathematics emerge from historical developments.

Concluding the series of case studies and shifting the focus to the 19th century, **Anna Kiel Steensen**'s chapter titled *Reading mathematical texts with structural semiotics* adopts the principles and tools of structuralist semiology, building on Wagner's presentation in this section, to conduct a *close reading* of a historical mathematical text. Steensen's proposes a proper operationalization of the commutation test, which enables her to identify semiotic units by examining substitutions within mathematical expressions and their corresponding contents. The specific text under scrutiny is Dedekind's *Über die Composition der binären quadratische Formen*, initially published in 1871. With exceptional precision, Steensen's analysis of this famous piece tackles the genesis of the mathematical notion of "ideal" from the internal perspective of the text.

Through the lens of structural semiotics, Steensen convincingly shows that the progressive specification of expressions within the text's narrative structure engenders a series of productive ambiguities requiring the intervention of the reader. The irreducible underspecification of certain signifiers thus revealed allows her to suggest two intriguing properties of mathematical signs which could be extended beyond this particular text. First, a mathematical text can—and sometimes must—be read before its content is rigorously established. Second, in this process, important dimensions of mathematical content are determined as the effect of the interaction between strictly expressive means. Looking ahead, while Steensen's analysis remains circumscribed to a single text, her contribution can have broader implications in the use of distributional analyses for the study of mathematical text and corpora.

The last two chapters of the section redirect the attention from historical analysis to current and future challenges in mathematical sign practices. In the first of these chapters, *The design of mathematical language*, **Jeremy Avigad** addresses the question of the *design choices* that arise when conceiving mathematics as a *semi-formal language*. Retrieving the theme of the previous chapter, Avigad observes that traditional formal languages used to represent mathematical content often face the issue of being underspecified in comparison to ordinary mathematical language. While the control and precision offered by formal methods are arguably essential for meeting the inferential requirements of mathematics, other features that are absent in formal specifications are equally necessary or desirable when considering mathematical language as the object of collective practices. This encompasses, among others, principles for effective error checking by a community, fostering creativity, ensuring efficiency, and maintaining reliability.

In addition to drawing on examples from the mathematical literature—particularly textbooks—Avigad's exploration of this problem crucially incorporates insights from the design of computational proof assistants and their libraries. This constitutes one of the principal originalities of Avigad's contribution, as it brings contemporary mathematical practices into play to reflect on the nature and conditions of mathematical language. While his approach is not necessarily limited to contemporary mathematics, Avigad's focus in this case centers around contemporary mathematical language. The outcome is an impressive survey of various features of mathematical language that are subject to design choices, ranging from the different categories used to organize mathematical objects to various abstraction principles, theory-building tools, and other aspects of mathematical language, such as diagrams, algorithms, and heuristics.

Avigad's *catalogue raisonné* is an insightful and useful tool for analyzing past practices and making informed decisions in current and future contexts. As computer-assisted mathematics gains prominence, Avigad's contribution provides essential guidance for effectively leveraging the benefits of formal methods while preserving the richness and flexibility of mathematical language as a dynamic and evolving tool for collective mathematical practice.

Finally, in *How to Do Maths With Words. Neural Machine Learning Applications to Mathematics and Their Philosophical Significance*, **Juan Luis Gastaldi** tackles one of the most pressing issues facing the mathematical community today—the widespread proliferation of *machine learning applications to mathematical knowledge*, stimulated by the development of Deep Neural Networks (DNNs) and Artificial Intelligence (AI) in the past decade. Gastaldi's primary goal is to demonstrate that these applications hold profound implications worthy of the attention and interest of historians and philosophers of mathematics, despite the natural skepticism that may arise toward current AI branding. Gastaldi bases his claim on two compelling reasons: the potential impact of these new technologies on mathematical practices and, more profoundly, the emergence of a new perspective on mathematical knowledge bearing a new and surprising relationship with the principles of natural language.

After providing a concise presentation of DNNs, Gastaldi conducts a meticulous survey of the work in this booming research area, unveiling its main identifiable orientations. Significantly, those trends tend to be spontaneously organized according to the AI researchers' implicit assumptions as to what particular practice characterizes the production of mathematical knowledge, namely: finding proofs, manipulating objects, acquiring skills, or supporting heuristics. Through this survey, Gastaldi highlights that despite the pronounced differences among various machine learning applications, they all share a distinct and novel perspective on mathematical language, at odds with the conventional view that understands mathematical expressions as simple notations for existing contents or as an arbitrary syntax for a predefined semantics. The reason is that the successful performance of tasks requiring the manipulation of mathematical content is grounded on the processing of textual data alone. Yet, this does not bring neural models closer to a formalist approach to mathematical knowledge and language, because the language implied both in the training data and in the learning models is no other than natural, informal language. Challenging a clear-cut distinction between natural and artificial languages, natural language datasets and models are thus mobilized to address tasks involving mathematical content in a way that no existing philosophy of mathematics could have foreseen.

The implications of this new role for natural language raise numerous open questions and challenges that demand attention from a pluralistic, critical, and historically informed theory of mathematical language—precisely like the one outlined in this entire section.

Inevitably, every publication comes with its share of *regrets*. Despite our best efforts, some approaches representing significant contributions to a practice-driven theory of mathematical signs and language could not find a place in this section. Four of them deserve to be briefly mentioned here, thus inviting the reader to explore these topics further and fill in the gaps left by their absence in this work. The first one concerns the study of mathematical language and other semiotic artifacts from the perspective of *ethnomathematics*. The work of Barton (2008) offers a beautiful example of what this approach can contribute to the study of mathematical language. The stimulating work of Vandendriessche (2015) also provides original insights in this regard. Second, it bearly deserves to be mentioned how much a philosophy of mathematical *notations* has to offer to a theory of mathematical signs. While Waszek's survey addresses some of this work, this section would have greatly benefited from a chapter dedicated to this important subject matter. We refer the interested reader to the references in Waszek's chapter. Thirdly, the intersection of *digital humanities*, *corpus linguistics*, and mathematical practices is an area of growing importance, where the automatic processing of mathematical texts plays a fundamental role. Insights from this research orientation can be particularly insightful to articulate many dimensions of semiological practices in mathematics. Fortunately, this Handbook features a chapter by Tanswell and Inglis (2020) providing an example of this interesting line of research. Finally, it is worth mentioning the combined efforts of the research communities gathered around the Conference on Intelligent Computer Mathematics (CICM). The results exhibited in the processing of many mathematical tasks assessed by this community—such as informal-formal translation, information retrieval, classification, formula parsing, OCR, digital edition, typesetting systems, automatic proof generation, etc—constitute valuable resources for the study of mathematical signs within the tradition of the history and philosophy of mathematical practice. We refer the reader to the series of proceedings of the Conference (CICM 2008-2022).

Conclusion

The culmination of our endeavor is a comprehensive work comprising over 250 pages, skillfully crafted by confirmed and emerging specialists in the field. We are confident that the collective efforts invested in this work will significantly enhance our understanding of the intricate connections between mathematical knowledge and semiological practices. By providing a diverse, consistent, and historically informed account of this important dimension of the history and philosophy of mathematical practices, we aspire to contribute to unraveling the distinctiveness of mathematics as a human practice. Ultimately, the aim of this work is to forge innovative pathways that bridge the realms of formal and natural sciences with the rich domain of the social sciences and the humanities, fostering a harmonious integration of knowledge across diverse disciplines.

Acknowledgements I want to thank Jeremy Avigad, Karine Chemla, Ladislav Kvasz, Reviel Netz, Anna Kiel Steensen, Lucien Vinciguerra, Roy Wagner and David Waszek for their generosity, patience, and invaluable contribution. I am deeply thankful to all the anonymous reviewers for their careful work, dedication, and constructive advice. Last but not least, I wish to express my gratitude to Bharath Sriraman for his constant support, comprehension and trust.

Funding Information This project has received funding from the *European Union's Horizon 2020 research and innovation programme* under grant agreement No 839730

References

Barthes R (1988) The Semiotic Challenge. Basil Blackwell, Oxford

Barton B (2008) The Language of Mathematics: Telling Mathematical Tales. Springer US, Boston, MA, DOI 10.1007/978-0-387-72859-9_1, URL `https://doi.org/10.1007/978-0-387-72859-9_1`

Chemla K (2004) History of science, history of text. Kluwer Academic, Boston

Chomsky N (1955) Logical syntax and semantics: Their linguistic relevance. Language 31(1):36–45

CICM (2008-2022) Intelligent computer mathematics. Springer International Publishing, Cham, URL `https://link.springer.com/conference/mkm`

Herreman A (2000) La topologie et ses signes: éléments pour une histoire sémiotique des mathématiques. L'Harmattan, Paris

Hjelmslev L (1975) Résumé of a Theory of Language. No. 16 in Travaux du Cercle linguistique de Copenhague, Nordisk Sprog-og Kulturforlag, Copenhagen

Hjelmslev L, Uldall HJ (1936) Synopsis of an outline of glossematics. London

Kvasz L (2008) Patterns of Change. Birkhäuser Basel

Locke J (2013) An Essay Concerning Human Understanding. The Clarendon Edition of the Works of John Locke, Oxford University Press, URL `http://oxfordscholarlyeditions.com/view/10.1093/actrade/9780198243861.book.1/actrade-9780198243861-book-1`

Macbeth D (2005) Frege's Logic. Harvard Univ. Press, includes bibliographical references (p. 183 - 197) and index

Netz R (1999) The Shaping of Deduction in Greek Mathematics. Cambridge University Press

Robering K, Posner R, Sebeok TA (eds) (1996) Semiotik, no. Bd. 13 in Handbücher zur Sprach- und Kommunikationswissenschaft, W. de Gruyter, URL `http://search.ebscohost.com/login.aspx?direct=true&scope=site&db=nlebk&db=nlabk&AN=281630`

Rotman B (1988) Toward a semiotics of mathematics. Semiotica 72(1-2):1–36, DOI doi:10.1515/semi.1988.72.1-2.1, URL `https://doi.org/10.1515/semi.1988.72.1-2.1`

Rotman B (2000) Mathematics as Sign: Writing, Imagining, Counting. Stanford University Press, Palo Alto

Russell B (2010) Principles of mathematics. Routledge classics, Routledge, London

Saussure Fd (1959) Course in General Linguistics. McGraw-Hill, New York, translated by Wade Baskin

Sebeok TA (2001) Signs: An introduction to semiotics, 2nd edn. Toronto studies in semiotics and communication, Univ. of Toronto Press

Serfati M (2005) La révolution symbolique: La constitution de l'écriture symbolique mathématique. Éditions Pétra, Paris

Sriraman B (ed) (2020) Handbook of the History and Philosophy of Mathematical Practice. Springer International Publishing, Cham, DOI 10.1007/978-3-030-19071-2, URL `https://doi.org/10.1007/978-3-030-19071-2`

Tanswell FS, Inglis M (2020) The Language of Proofs: A Philosophical Corpus Linguistics Study of Instructions and Imperatives in Mathematical Texts, pp 1–28. In: Sriraman (2020), DOI 10.1007/978-3-030-19071-2_50-1, URL `https://doi.org/10.1007/978-3-030-19071-2_50-1`

Vandendriessche E (2015) String figures as mathematics? : an anthropological approach to string figure-making in oral tradition societies. Springer, Cham

Vinciguerra L (1999) Langage, visibilité, différence: histoire du discours mathématique de l'âge classique au XIXe siècle. J. Vrin, Paris