PhilML'23
Eberhard Karls University of Tübingen
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The Language of Mathematics

Epistemological Consequences of the Application of Neural Models to Mathematics

Juan Luis Gastaldi

ETH zürich

September 13, 2023

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European Union's Horizon 2020 research and innovation programme
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Reference Papers

- Gastaldi, J. L., & Pellissier, L. (2021). The calculus of language: Explicit representation of emergent linguistic structure through type-theoretical paradigms. *Interdisciplinary Science Reviews*. https://doi.org/10.1080/03080188.2021.1890484

- Gastaldi, J. L. (Forthcoming 2004b). How to Do Maths with Words. Neural Machine Learning Applications to Mathematics and Their Philosophical Significance. In B. Sriraman (Ed.), *Handbook of the History and Philosophy of Mathematical Practice*. Springer

- Bradley, T.-D., Gastaldi, J. L., & Terilla, J. (Forthcoming 2024a). The structure of meaning in language: parallel narratives in linear algebra and category theory. *Notices of the AMS*

- Gastaldi, J. L. (Forthcoming 2024c). Content from Expressions. The Place of Textuality in Deep Learning Approaches to Mathematics. *Synthese (under review)*

Outline

Neural ML and Mathematics

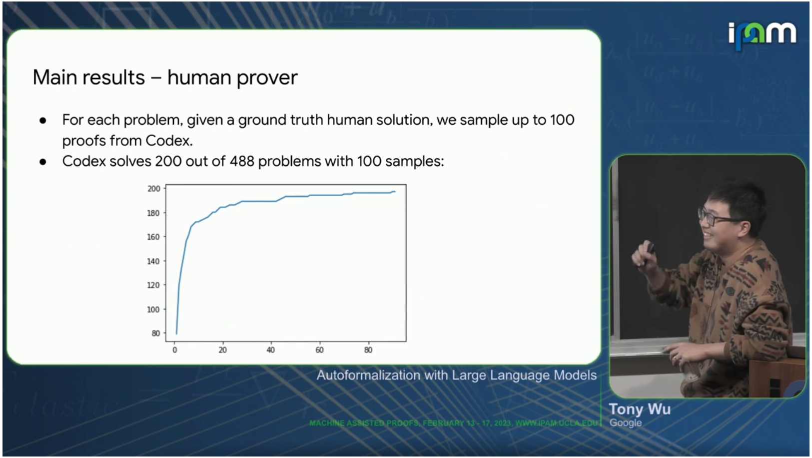
Epistemology of Natural Language Processing (NLP)

Distributional Arithmetic

Takeaways

Outline

Tony Wu at the IPAM



Tony Wu, *Autoformalization with Large Language Models* (IPAM (UCLA), Feb 15, 2023)

Melanie Mitchell on PaLM2

Melanie Mitchell
@MelMitchell1

Weird statement from Google's Palm 2 announcement.

(from blog.google/technology/ai/...)

- **Reasoning:** PaLM 2's wide-ranging dataset includes scientific papers and web pages that contain mathematical expressions. As a result, it demonstrates improved capabilities in logic, common sense reasoning, and mathematics.

6:23 PM · May 12, 2023 · **52.9K** Views

https://blog.google/technology/ai/google-palm-2-ai-large-language-model/

What’s So Funny?

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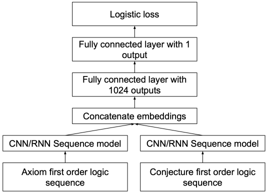
- Mathematical competence was supposed to be a condition to write and read scientific papers and mathematical expressions instead of their miraculous effect.
- Natural language was considered the cause of rather than the solution to the multiple problems preventing mathematics from achieving higher degrees of precision.
- More generally, the *formal* nature of mathematics was believed to make it impassive to the strong empirical position assumed by connectionist approaches guiding the application of DNNs.

Main Orientations in DNN Applications to Maths

(Gastaldi, Forthcoming 2004b)

- Proof-Oriented
  - Bansal et al., 2019; Polu and Sutskever, 2020; Wu et al., 2022.



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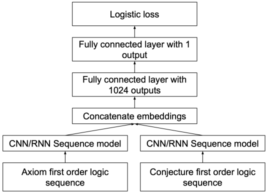

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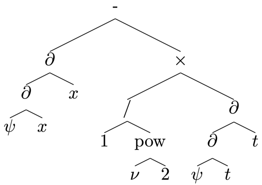

(Lample and Charton, 2019)

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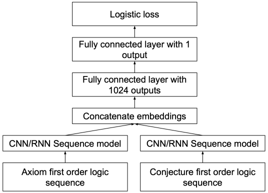

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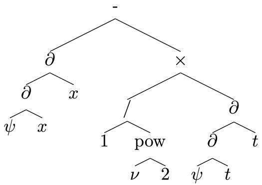
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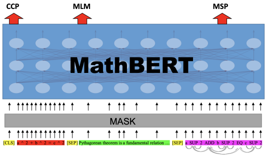

(Peng et al., 2021)

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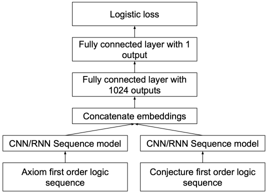
  - Brown et al., 2020; Lewkowycz et al., 2022; Shen et al., 2021
- Heuristic-Oriented
  - Davies et al., 2021; Wagner, 2021



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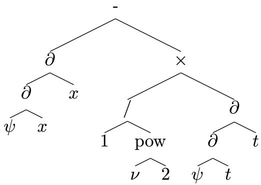

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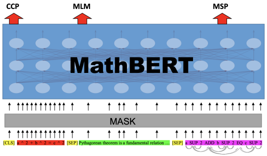

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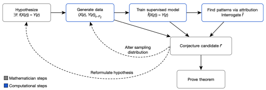

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- Research orientations tend to be spontaneously organized according to the AI researchers' implicit assumptions as to *what it is that we do when we do mathematics*.
- However, practically all applications share a common philosophical assumption: *Natural language* plays a critical role in the processing mathematical knowledge.
- The potential success of DNN methods in mathematics is inseparable from a reorientation of the epistemology of mathematics from logic and formal systems to natural language practice.

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Stochastic Parrots vs. AI Consciousness

📉🦜

Language models are not like us,
therefore they do not and can not have any relation to meaning.

🤖🧠

Language models have a relation to meaning,
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Language models have a relation to meaning,
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- Two tools to resist to this alternative:
  - Conceptual: Operational notion of *formal content*
  - Technical: Distributionalism through *algebraic word vector representations*

Formal Content

(Gastaldi and Pellissier, 2021;Gastaldi, Forthcoming 2024c)

Form ~~vs.~~ and ~~Meaning~~ Content

Kant, Hegel, Frege, Saussure, Hjelmslev, etc.

Formal Content: The dimension of content which finds its source in the internal relations holding between the expressions of a language

- *Syntactic Content*: The content a unit receives as a result of the multiple dependencies it can maintain with respect to other units in its context
- *Characteristic Content*: The content resulting from the inclusion of a unit in a class of other units by which it accepts to be substituted in given contexts
- *Informational Content*: The content related to the non-uniform distribution of units within those substitutability classes

Illustration of Formal Contents

(Gastaldi and Pellissier, 2021;Gastaldi, Forthcoming 2024c)

Characteristic Content

{cat, dog, spider, gavagai}

Clustering

Class

Syntactic Content

"the gavagai is on the mat"

Type Theory

Type

Informational Content

{cat:0.059%, dog:0.012%, spider:0.009% gavagai:0.000%}

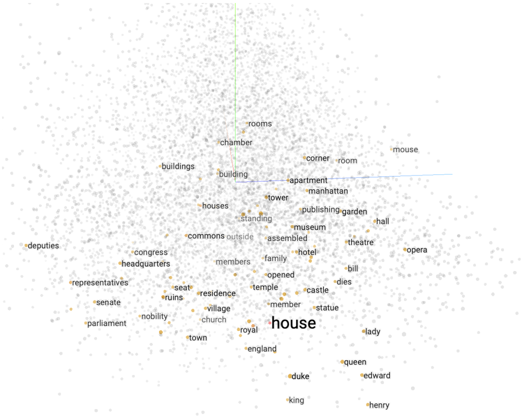
Probability and Information Theory

Probability Distribution

Distributionalism and Word Embeddings

- Distributional Hypothesis (Harris, 1960; Saussure, 1959)
  - "You shall know a word by the company it keeps!" (Firth, 1935)
  - The content of a linguistic unit is determined by its distribution over a corpus (i.e., the other units appearing in its context)

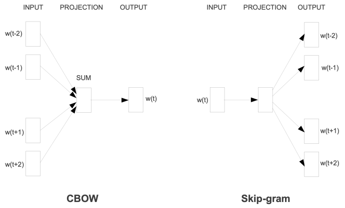
- Computational interpretation: Word Embeddings

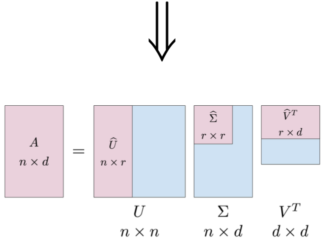


(https://projector.tensorflow.org)

Word Embeddings as Matrix Factorization

- Word2vec performs an implicit factorization of a word-context matrix (Levy and Goldberg, 2014)
  - (shifted) pointwise mutual information (PMI)
  - Truncated SVD to reduce dimensionality





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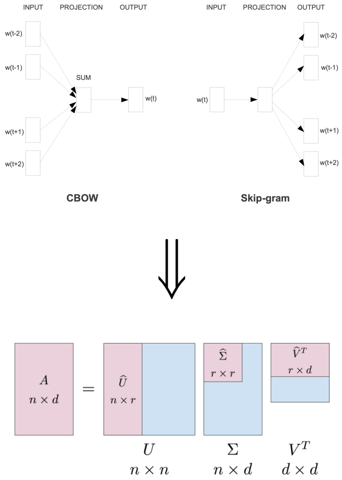
- Equivalent results can be achieved with explicit vector representations (Levy et al., 2015)



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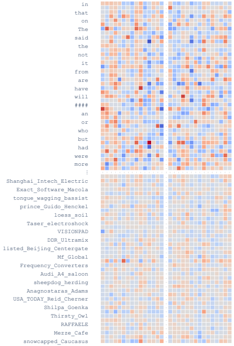

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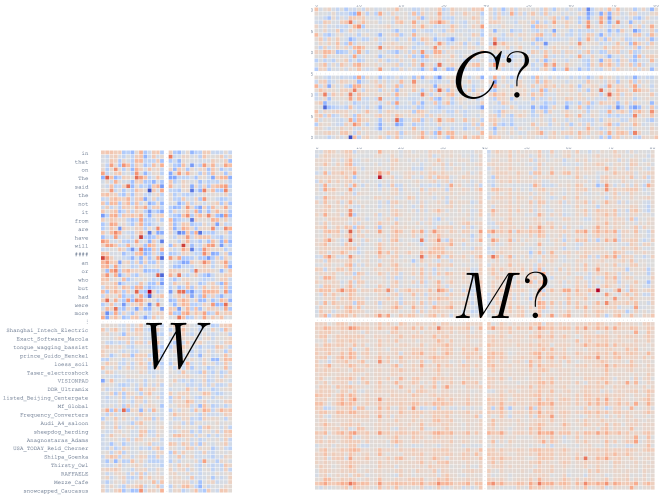

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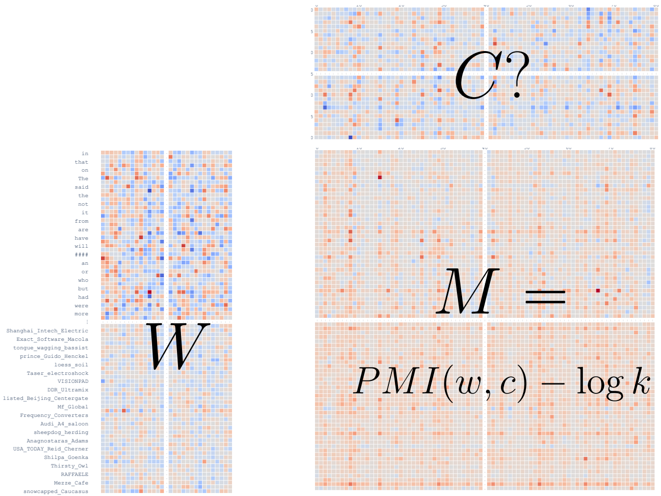

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$$PMI(w,c) = \log \frac{p(w,c)}{p(w)p(c)}$$

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Arithmetical Content

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Recursion through Peano Axioms

1. 0 is a number.

2. If n is a number, the successor of n is a number.

3. 0 is not the successor of a number.

4. Two numbers of which the successors are equal are themselves equal.

5. If a set $\mathbf{S}$ of numbers contains 0 and also the successor of every number in $\mathbf{S}$, then every number is in $\mathbf{S}$ (induction axiom).

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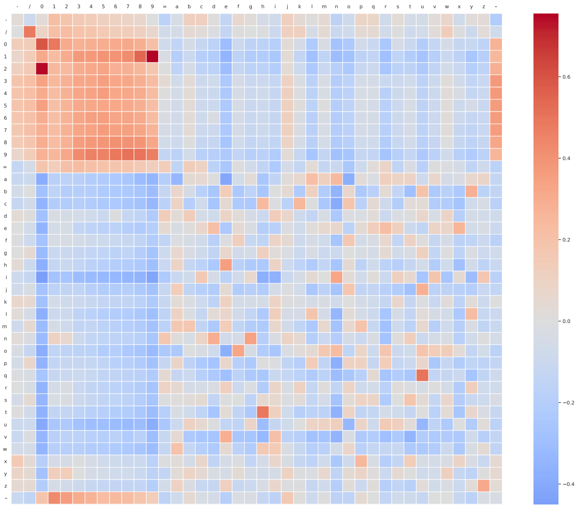
The Distributional Properties of Characters

$$A_{i,j} = pmi(c_i; c_j) = \log \frac{p(c_i, c_j)}{p(c_i)p(c_j)}$$



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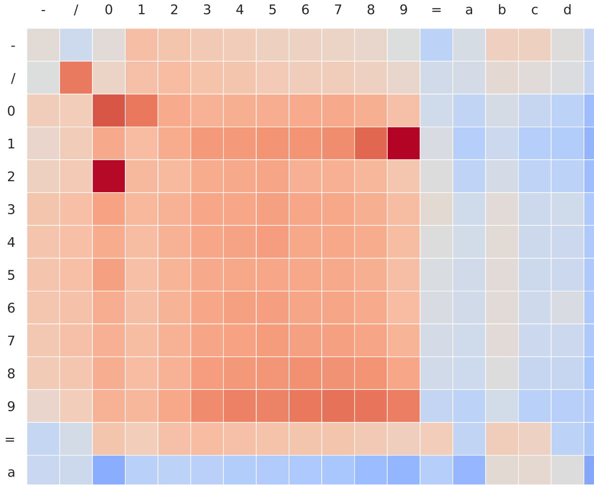

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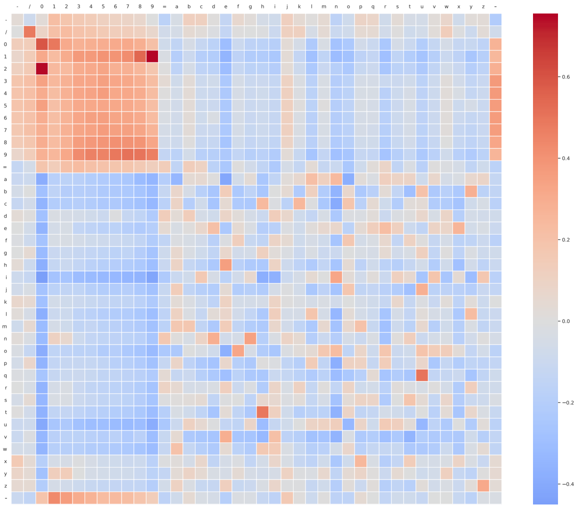
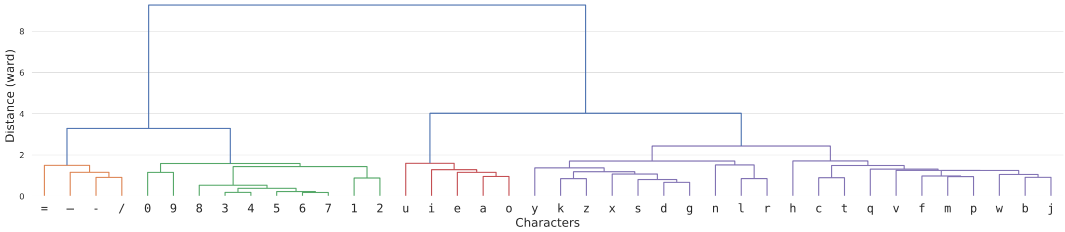


The Characteristic Content of Digits



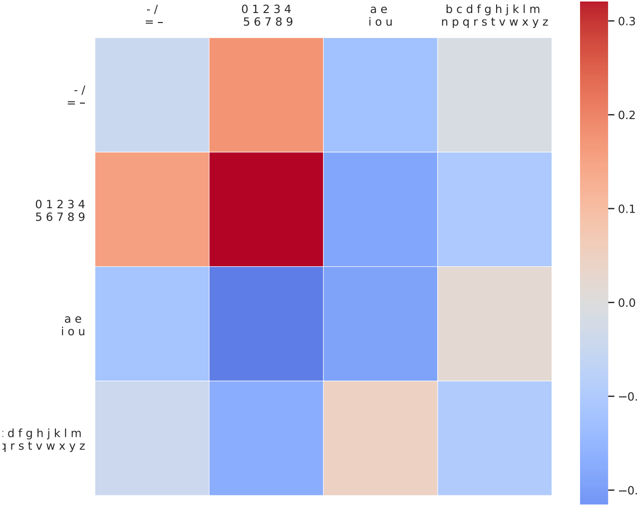
$$O := \{=, -, \text{-}, /\}$$

$$D := \{0, 9, 8, 3, 4, 5, 6, 7, 1, 2\}$$

$$V := \{\mathtt{u}, \mathtt{i}, \mathtt{e}, \mathtt{a}, \mathtt{o}\}$$

$$C := \{\mathtt{y}, \mathtt{k}, \mathtt{z}, \mathtt{x}, \mathtt{s}, \mathtt{d}, \mathtt{g}, \mathtt{n}, \mathtt{l}, \mathtt{r}, \mathtt{h}, \mathtt{c}, \mathtt{t}, \mathtt{q}, \mathtt{v}, \mathtt{f}, \mathtt{m}, \mathtt{p}, \mathtt{w}, \mathtt{b}, \mathtt{j}\}$$

The Syntactic Content of Digits



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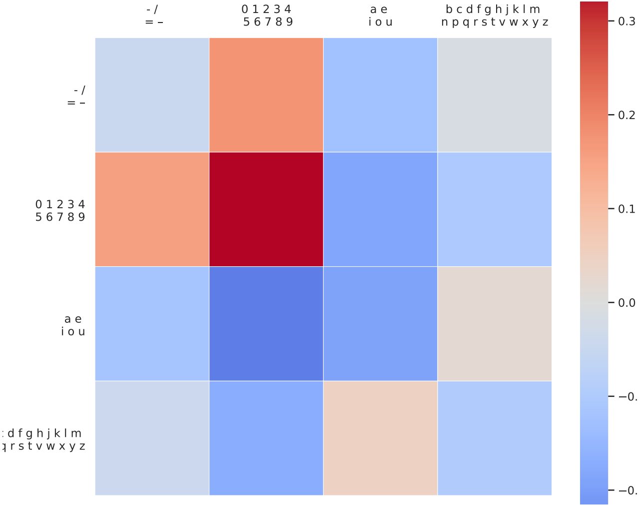

$$f(c_n) = c_{n+1}$$

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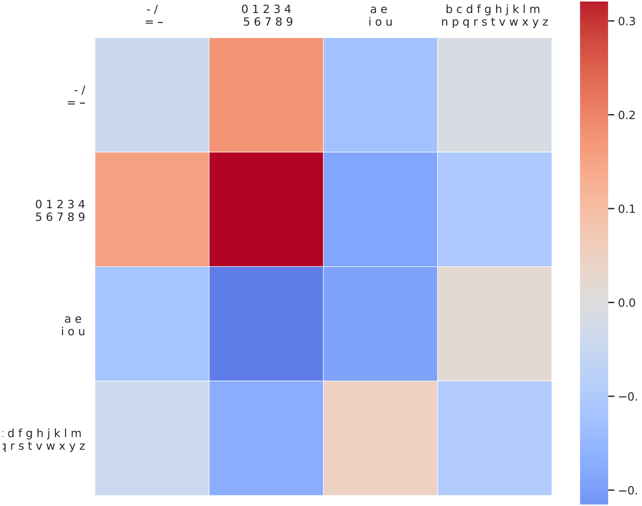

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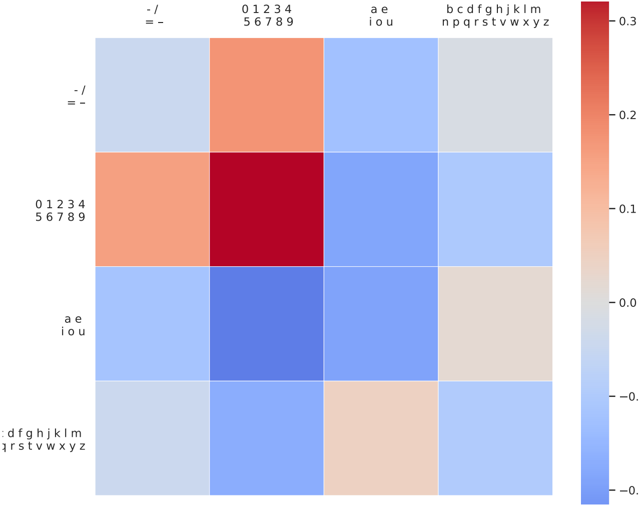


$$f(c_n) = c_{n+1}$$

$$f(D) = D$$

$$f(D + \mathtt{d}_0) = D + \mathtt{d}_1$$

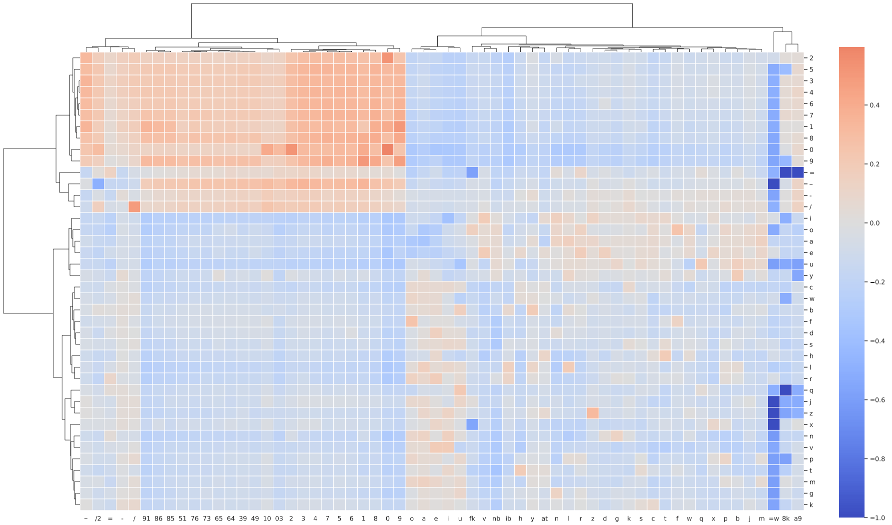
$$f = T \circ t$$

$$T(D) = D$$

Concatenation and Characteristic Content of $\mathbb{N}$

$$
\begin{aligned}
&D \times f(D) \\
&= D \times D
\end{aligned}
$$

Concatenation and Characteristic Content of ℕ



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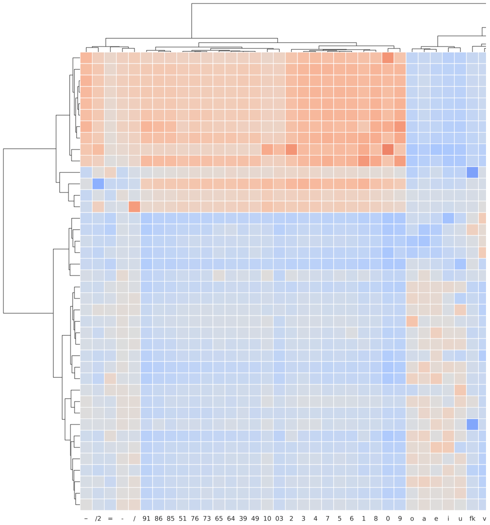

$$D \times f(D)$$
$$= D \times D$$

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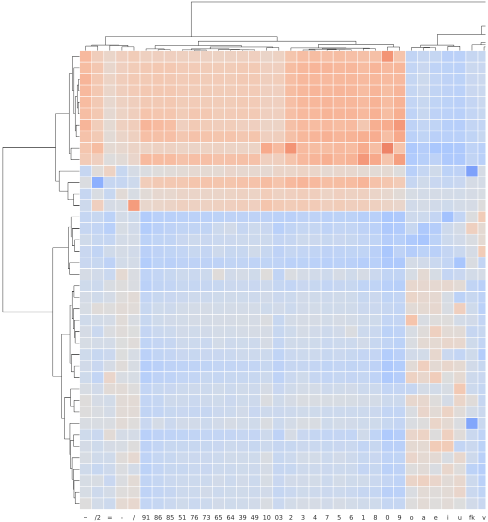

$$D \times f(D)$$
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$$f(\overrightarrow{D \times D}) \approx D$$

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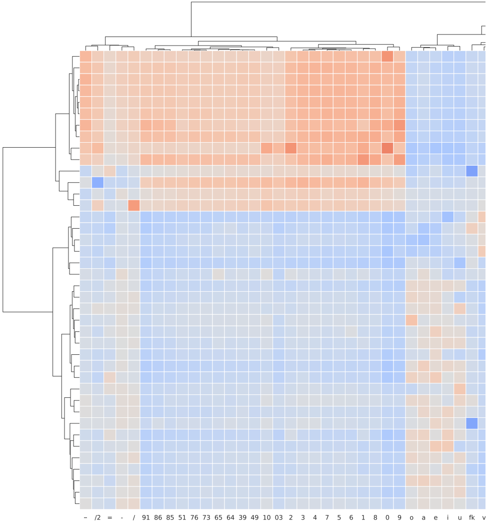

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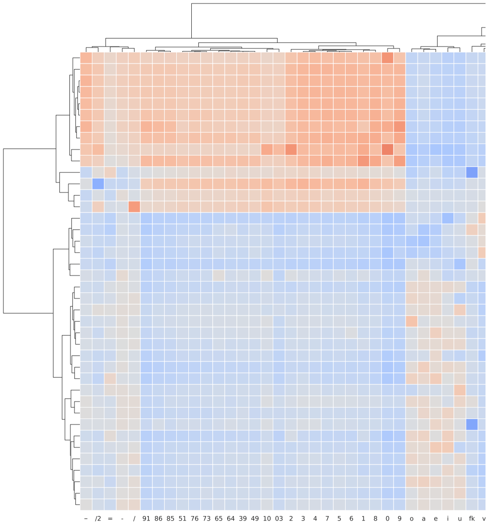

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$$f_2(D^2) = D^2$$

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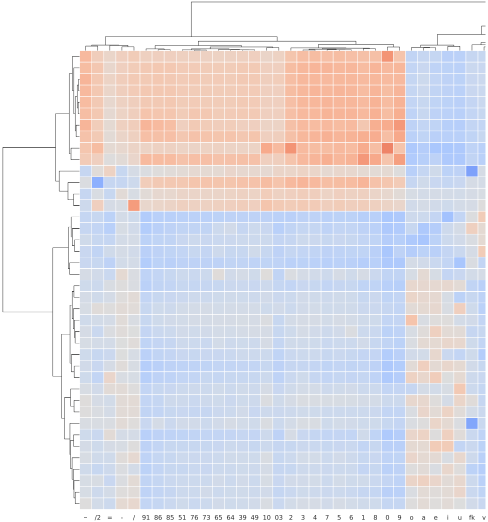

$$D \times f(D)$$
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$$f(\overrightarrow{D \times D}) \approx D$$

$$D^2 = (D \times D) \cup D$$

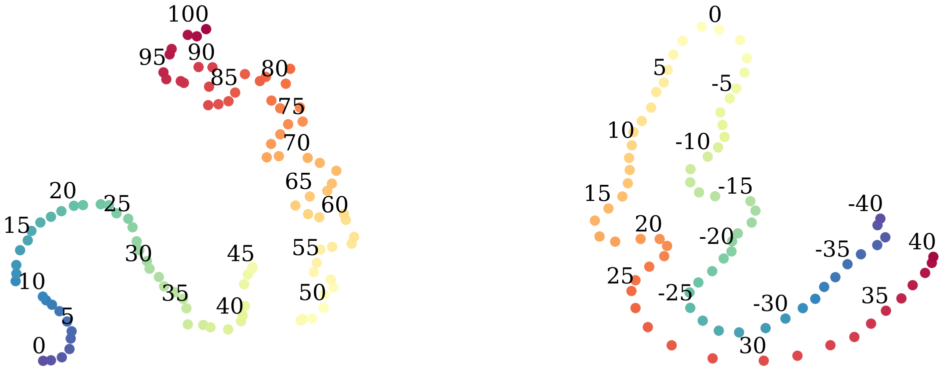
$$f_2(D^2) = D^2$$

$$\vdots$$

$$f_*(D^*) = D^*$$

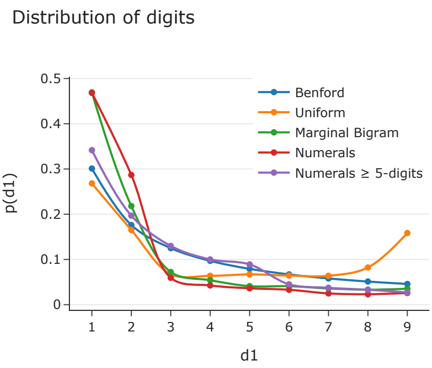
$$\mathbb{N} = D^* \approx D^n$$

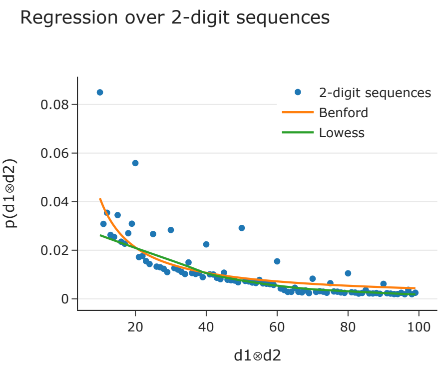
Number Embeddings



t-SNE of the embeddings of the integer model (left) and
the exponent embeddings of the float model (right) in d'Ascoli et al., 2022.

Total Order Through Benford's Law





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Takeaways

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- Epistemology of neural ML applications in science through case studies rather than general speculative principles.

- In the case of mathematics: New and original role of natural language.

- Remove the magic showing how what was assumed to be impossible is possible in principle and to some degree independent of neural techniques.

- The hope is that this perspective at the crossroads of many disciplines (mathematics, linguistics, ML) can provide epistemological insights to all of them:
  - Maths: Distributional account of mathematical objects.
  - NLP: Reorientation of philosophical debate, away from 📉🦜 vs 🤖🧠 debate.
  - ML: Interpretability tools and new links between statistical and symbolic features.

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